

An Invariant Charge Model for All $q^2 > 0$ in QCD and Gluon Condensate

A.I. Alekseev*

Institute for High Energy Physics, 142281 Protvino, Moscow Region, Russia

B.A. Arbuzov†

Skobeltsyn Institute for Nuclear Physics, Moscow State University
119899 Moscow, Russia

Abstract

Under assumption of singular behavior of invariant charge $\alpha_s(q^2)$ at $q^2 \simeq 0$ and of large q^2 behavior, corresponding to the perturbation theory up to four loops, a procedure is considered of smooth matching the β -function at a boundary of perturbative and nonperturbative regions. The procedure results in a model for α_s for all $q^2 > 0$ with dimensionless parameters being fixed and dimensional parameters being expressed in terms of only one quantity Λ_{QCD} . The gluon condensate which is defined by the nonperturbative part of the invariant charge is calculated for two variants of “true perturbative” invariant charge, corresponding to freezing option and to analytic one in nonperturbative region. Dimensional parameters are fixed by varying normalization condition $\alpha_s(m_\tau^2) = 0.29, 0.30, \dots, 0.36$. It is obtained that on the boundary of perturbative region $\alpha_0 = \alpha_s(q_0^2) \simeq 0.44$, the procedure results in nonperturbative Coulomb component $\alpha_{Coulomb} \simeq 0.25$, the nonperturbative region scale $q_0 \simeq 1$ GeV, the model parameter $\sigma \simeq (0.42 \text{ GeV})^2$ which suits as string tension parameter, the gluon condensate appears to be close for two variants considered, $K \simeq (0.33 - 0.36 \text{ GeV})^4$ (for $\alpha_s(m_\tau^2) = 0.33$).

Keywords: Nonperturbative QCD; gluon condensate; running coupling constant; infrared region.

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*E-mail: alekseev@mx.ihep.su

†E-mail: arbuzov@theory.sinp.msu.ru

1 Introduction

A consistent description of interaction of fundamental fields both at large and short distances is one of the most important problems of QCD. The strength of the interaction is defined by invariant charge $\alpha_s(q^2)$ (the running coupling constant), which satisfies the renormalization group equation. The purpose of the present work consists in a formulation of a model for $\alpha_s(q^2)$ for all $q^2 > 0$, which is appropriate for description both of perturbative and of nonperturbative phenomena and needs minimal number of parameters.¹

We assume that there exists some value q_0 , which characterizes the non-perturbative effects scale and the corresponding value $\alpha_0 = \alpha_s(q_0^2)$ such, that for $q^2 > q_0^2$ and thus $\alpha_s(q^2) < \alpha_0$ the finite loop perturbation theory is applicable and sufficient, while nonperturbative effects prevails in region $q^2 < q_0^2$, which contains also nonphysical singularities of the perturbation theory and so here this theory essentially needs an extension of definition. In general, the invariant charge and the beta-function as well are depending on a renormalization scheme [2, 3]. For definiteness while performing calculations at $q^2 \geq q_0^2$ we use \overline{MS} scheme. It is well-known, that the β -function in the perturbative QCD is of the form

$$\beta_{\text{pert}}(h) = -b_0 h^2 - 2b_1 h^3 - \frac{b_2}{2} h^4 - b_3 h^5 + O(h^6), \quad h = \frac{\alpha_s(q^2)}{4\pi}. \quad (1)$$

For $n_f = 3$ we have values of coefficients $b_0 = 9$, $b_1 = 32$, $b_2 \simeq 1287.67$, $b_3 \simeq 12090.38$ (coefficients b_0 , b_1 do not depend on renormalization scheme while values b_2 , b_3 correspond to a choice of \overline{MS} -scheme). Expressions obtained by solution of Gell-Mann – Low equation

$$q^2 \frac{\partial h(q^2)}{\partial q^2} = \beta(h) \quad (2)$$

with the use of (1), are widely used for sufficiently large momenta transfer, however they can not be applied in the infrared region.

As a matter of fact the behavior of α_s at small momenta till now is an open question. Lattice methods and SD equations give no an ultimate answer. For behavior of the invariant charge $\alpha_s(q^2)$ at $q^2 \rightarrow 0$ a number of variants are considered (see, e.g. Ref. [4]). Now the most popular variants for $q^2 \rightarrow 0$ behavior are: $\alpha_s \rightarrow 0$, $\alpha_s \rightarrow O(1)$, α_s is strongly enhanced. We consider the last possibility, in particular the well-known singular infrared

¹This paper is an extended version of Ref. [1].

asymptotic behavior

$$\alpha_s(q^2) \simeq \frac{g^2}{4\pi} \frac{M^2}{q^2}, \quad q^2 \rightarrow 0, \quad (3)$$

for a review see, e.g. Ref. [5] and more recent papers Refs. [6]. Such behavior of the invariant charge α_V in so-called V-scheme corresponds to a linear confining quark-antiquark static potential at that $g^2 M^2 = 6\pi\sigma$, where σ is the string tension. Results of some works [7] on the lattice study of the three-gluon vertex demonstrate a necessity of taking into account of non-perturbative contributions to the running coupling constant being of the form (3). In the framework of the continuous QFT additional arguments in favor of behavior (3) are also presented in paper [8]. Highly similar singular infrared behavior has recently developed model for the QCD analytic invariant charge [9].

Asymptotic behavior (3) occurs provided $\beta(h) \rightarrow -h$ for $h \rightarrow \infty$. We consider a possibility of behavior (3) and assume the following form of the infrared β -function

$$\beta(h) = -h + z, \quad h > h_0, \quad (4)$$

where z is a constant and h_0 corresponds to the boundary between perturbative and nonperturbative regions. For $h < h_0$ we shall use β -function (1) with a finite number of loops taken into account. Our recipe for construction of β -function for all $h > 0$ consists in a smooth matching of expressions (4) and (1) at point $h = h_0$ in approximations of the perturbation theory up to four loops. The demand of the β -function and its derivative to be continuous uniquely fix free parameters z and h_0 of the global β -function (the matched one). Note, that the presence of parameter z in Eq. (4) which corresponds to Coulomb contribution in invariant charge gives a possibility of smooth matching.

We are willing to build the model for the invariant charge, which precisely coincides the perturbation theory in the perturbative region $q^2 > q_0^2$, while in the nonperturbative infrared region it provides simple description of main nonperturbative parameters. The work is organized in the following way. In Section 2 we obtain the matched solutions for the cases of 1 — 4 loops. Dimensionless parameters of the model are uniquely defined and further the solutions are normalized at the scale of the τ -lepton mass, that leads to definite values of dimensional parameters Λ , q_0 , σ . In Section 3 the gluon condensate is calculated. In doing this we consider first a possibility of freezing of the perturbative component of α_s in the infrared region and secondly a possibility of analytic behaviour of this component in the infrared region. Section 4 contains concluding remarks.

2 Construction of One — Four-Loop Matched Solutions for All $q^2 > 0$

For an illustration let us consider the most simple one-loop case. Conditions of matching give two equations

$$\begin{aligned} -b_0 h_0^2 &= -h_0 + z, \\ -2b_0 h_0 &= -1. \end{aligned} \tag{5}$$

The solution of set (5) reads

$$h_0 = \frac{1}{2b_0}, \quad z = \frac{1}{4b_0}. \tag{6}$$

We shall normalize perturbative solution

$$\alpha_s(q^2) = \frac{4\pi}{b_0 \ln x}, \quad x = \frac{q^2}{\Lambda_{QCD}^2}, \quad q^2 \geq q_0^2 \tag{7}$$

by value $4\pi h_0$, that gives

$$x_0 = q_0^2 / \Lambda_{QCD}^2 = e^2, \tag{8}$$

where $e = 2.71828\dots$. Imposing on $\alpha_s(q^2)$ the natural condition to be continuous at $q^2 = q_0^2$, we may normalize nonperturbative solution of equation (2)

$$\alpha_s(q^2) = 4\pi \left(\frac{C}{q^2} + z \right), \quad q^2 \leq q_0^2 \tag{9}$$

by $4\pi h_0$ as well. As a result we obtain

$$C = \frac{3\sigma}{8\pi} = q_0^2(h_0 - z), \quad c_0 \equiv C/\Lambda_{QCD}^2 = x_0(h_0 - z). \tag{10}$$

Equations (10) are correct for one — four loops, for one-loop case $c_0 = e^2/4b_0$. For final fixation of the solution for all $q^2 > 0$ we need to define Λ_{QCD} by normalizing the solution, say, at point $q^2 = m_\tau^2$, where m_τ is the mass of the τ -lepton.

For one-loop case we have following simple formulae for quantities under consideration

$$\alpha_0 = 2\pi/b_0, \quad \alpha_{\text{Coulomb}} = \pi/b_0,$$

$$q_0 = e\Lambda_{QCD}, \quad \sigma = \frac{2\pi e^2}{3b_0}\Lambda_{QCD}^2, \quad K = \frac{3e^4}{2\pi^2 b_0}\Lambda_{QCD}^4. \quad (11)$$

The gluon condensate K for frozen perturbative constituent which is given here for completeness will be calculated below. For $n_f = 3$ one has $\alpha_0 = 0.698$, $\alpha_{\text{Coulomb}} = 0.349$, $q_0 = 2.72\Lambda_{QCD}$, $\sigma = (1.31\Lambda_{QCD})^2$, $K = (0.980\Lambda_{QCD})^4$. For normalization condition, e.g. $\alpha_s(m_\tau^2) = 0.32$ one obtains $\Lambda_{QCD} = 0.201$ GeV and we have only qualitative agreement for quantities considered. As we shall see further on, rather poor results in the one-loop case are marginal, while results for many-loop cases turn to be much more promising.

Let us consider multi-loop cases. Solution $h(q^2)$ of equation (2) for $L = \ln(q^2/\Lambda^2) \rightarrow \infty$ reads as follows

$$\begin{aligned} h(q^2) = \frac{1}{b_0 L} \left\{ 1 - \frac{2b_1}{b_0^2 L} \ln L + \frac{4b_1^2}{b_0^4 L^2} \left[\ln^2 L - \ln L - 1 + \frac{b_0 b_2}{8b_1^2} \right] \right. \\ \left. - \frac{8b_1^3}{b_0^6 L^3} \left[\ln^3 L - \frac{5}{2} \ln^2 L - \left(2 - \frac{3b_0 b_2}{8b_1^2} \right) \ln L \right. \right. \\ \left. \left. + \frac{1}{2} - \frac{b_0^2 b_3}{16b_1^3} \right] + O\left(\frac{1}{L^4}\right) \right\}. \end{aligned} \quad (12)$$

Keeping in the expression terms with powers of logarithms in denominators up to the first, the second, the third and the fourth, we fix the 1 — 4-loop approximations of the perturbation theory for running coupling constant. It may be written in the form

$$\begin{aligned} \alpha_s(q^2) = 4\pi h(q^2) = \frac{4\pi}{b_0} a(x), \\ a(x) = \frac{1}{\ln x} - b \frac{\ln(\ln x)}{\ln^2 x} + b^2 \left[\frac{\ln^2(\ln x)}{\ln^3 x} - \frac{\ln(\ln x)}{\ln^3 x} + \frac{\kappa}{\ln^3 x} \right] \\ - b^3 \left[\frac{\ln^3(\ln x)}{\ln^4 x} - \frac{5}{2} \frac{\ln^2(\ln x)}{\ln^4 x} + (3\kappa + 1) \frac{\ln(\ln x)}{\ln^4 x} + \frac{\bar{\kappa}}{\ln^4 x} \right]. \end{aligned} \quad (13)$$

Here $x = q^2/\Lambda^2$, and coefficient are defined as follows

$$\begin{aligned} b &= \frac{2b_1}{b_0^2}, \\ \kappa &= -1 + \frac{b_0 b_2}{8b_1^2}, \\ \bar{\kappa} &= \frac{1}{2} - \frac{b_0^2 b_3}{16b_1^3}. \end{aligned} \quad (14)$$

Coefficients b , κ , $\bar{\kappa}$ depend on n_f . With $n_f = 3$ we have $b \simeq 0.7901$, $\kappa \simeq 0.4147$, $\bar{\kappa} \simeq -1.3679$. In the case of the two-loop approximation for perturbative α_s we have the following set of equations

$$\begin{aligned} b_0 h_0^2 + 2b_1 h_0^3 &= h_0 - z, \\ 2b_0 h_0 + 6b_1 h_0^2 &= 1, \\ \frac{1}{\ln x_0} - b \frac{\ln(\ln x_0)}{\ln^2 x_0} &= b_0 h_0, \\ \frac{c_0}{x_0} + z &= h_0. \end{aligned} \tag{15}$$

The set for three loops reads

$$\begin{aligned} b_0 h_0^2 + 2b_1 h_0^3 + \frac{b_2}{2} h_0^4 &= h_0 - z, \\ 2b_0 h_0 + 6b_1 h_0^2 + 2b_2 h_0^3 &= 1, \\ \frac{1}{\ln x_0} - b \frac{\ln(\ln x_0)}{\ln^2 x_0} + b^2 \left[\frac{\ln^2(\ln x_0)}{\ln^3 x_0} - \frac{\ln(\ln x_0)}{\ln^3 x_0} + \frac{\kappa}{\ln^3 x_0} \right] &= b_0 h_0, \\ \frac{c_0}{x_0} + z &= h_0. \end{aligned} \tag{16}$$

The set for four loops reads

$$\begin{aligned} b_0 h_0^2 + 2b_1 h_0^3 + \frac{b_2}{2} h_0^4 + b_3 h_0^5 &= h_0 - z, \\ 2b_0 h_0 + 6b_1 h_0^2 + 2b_2 h_0^3 + 5b_3 h_0^4 &= 1, \\ \frac{1}{\ln x_0} - b \frac{\ln(\ln x_0)}{\ln^2 x_0} + b^2 \left[\frac{\ln^2(\ln x_0)}{\ln^3 x_0} - \frac{\ln(\ln x_0)}{\ln^3 x_0} + \frac{\kappa}{\ln^3 x_0} \right] & \\ -b^3 \left[\frac{\ln^3(\ln x_0)}{\ln^4 x_0} - \frac{5}{2} \frac{\ln^2(\ln x_0)}{\ln^4 x_0} + (3\kappa + 1) \frac{\ln(\ln x_0)}{\ln^4 x_0} + \frac{\bar{\kappa}}{\ln^4 x_0} \right] &= b_0 h_0, \\ \frac{c_0}{x_0} + z &= h_0. \end{aligned} \tag{17}$$

From sets of equations (15) – (17) we find values of h_0 , z , x_0 and c_0 . They are presented in Table 1. Taking into account the existing data [10, 11, 12] we finally fix the momentum dependence of solutions for a number of values of the running coupling constant at τ -lepton mass scale m_τ with the effective number of flavors $n_f = 3$. The values of Λ_{QCD} corresponding to these normalization conditions are presented in Table 2, values of boundary momentum $q_0 = \sqrt{x_0} \Lambda_{QCD}$ are presented in Table 3, the string tension parameter $\sigma = (8\pi c_0/3) \Lambda_{QCD}^2$ is given in Table 4.

Table 1: The dimensionless parameters $h_0, z, x_0 = q_0^2/\Lambda_{QCD}^2, c_0 = C/\Lambda_{QCD}^2$ on the number of loops, $n_f = 3$.

	1-loop	2-loop	3-loop	4-loop
h_0	0.0556	0.0392	0.0356	0.0337
z	0.0278	0.0215	0.0203	0.0197
x_0	7.3891	7.7763	10.2622	12.4305
c_0	0.2053	0.1374	0.1572	0.1741

Table 2: Values of the parameter Λ_{QCD} (GeV) on loop numbers and normalization conditions. Normalization conditions: $\alpha_s(m_\tau^2) = 0.29, 0.30, \dots, 0.36$ with $m_\tau = 1.77703$ GeV, $n_f = 3$.

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.1600	0.3168	0.2873	0.2837
0.30	0.1734	0.3370	0.3069	0.3026
0.31	0.1869	0.3568	0.3263	0.3212
0.32	0.2005	0.3762	0.3454	0.3394
0.33	0.2143	0.3951	0.3642	0.3573
0.34	0.2280	0.4136	0.3827	0.3749
0.35	0.2418	0.4315	0.4007	0.3920
0.36	0.2556	0.4490	0.4184	0.4087

3 Gluon Condensate

Let us turn to calculation of the gluon condensate. Its value is defined by the nonperturbative part of α_s . We have (see, e.g. the third of Refs. [6])

$$K \equiv \langle \alpha_s/\pi : G_{\mu\nu}^a G_{\mu\nu}^a : \rangle = \frac{3}{\pi^3} \int_0^\infty dq^2 q^2 \alpha_{\text{npt}}(q^2). \quad (18)$$

We define the nonperturbative part of the invariant charge as a difference of total invariant charge α_s and its perturbative part (to all orders). In our approach the nonperturbative contributions are presented for $q^2 < q_0^2$ only.

Table 3: Values of the parameter q_0 (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2.

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.4350	0.8833	0.9203	1.0002
0.30	0.4713	0.9397	0.9831	1.0667
0.31	0.5081	0.9949	1.0452	1.1323
0.32	0.5451	1.0490	1.1065	1.1967
0.33	0.5824	1.1018	1.1667	1.2598
0.34	0.6198	1.1533	1.2258	1.3216
0.35	0.6572	1.2034	1.2838	1.3820
0.36	0.6947	1.2522	1.3405	1.4409

So we have

$$K = \frac{3}{\pi^3} \int_0^\infty dq^2 q^2 (\alpha_s(q^2) - \alpha_{\text{pt}}(q^2)) = \frac{3}{\pi^3} \int_0^{q_0^2} dq^2 q^2 (\alpha_s(q^2) - \alpha_{\text{pt}}(q^2)). \quad (19)$$

Let us consider two variants of the invariant charge perturbative component behaviour.

3.1 Freezing of perturbative component

For the beginning we define the perturbative part in nonperturbative region basing on the assumption of freezing of α_{pt} at small q^2 (see, e.g. Ref. [13]). That is we assume

$$\alpha_{\text{pt}}(q^2) = \alpha_s(q_0^2) = 4\pi h_0, \quad q^2 < q_0^2. \quad (20)$$

Using expressions (9), (20), we have

$$\begin{aligned} K_{\text{fr}} &= \frac{12}{\pi^2} \int_0^{q_0^2} dq^2 q^2 \left(\frac{C}{q^2} + z - h_0 \right) = \frac{12}{\pi^2} q_0^4 \left(\frac{C}{q_0^2} + \frac{1}{2}(z - h_0) \right) \\ &= \frac{6}{\pi^2} (h_0 - z) x_0^2 \Lambda_{QCD}^4. \end{aligned} \quad (21)$$

Expression (21) is valid for each of 1 — 4-loop approximations of α_s in perturbative region and ratio $K_{\text{fr}}/\Lambda_{QCD}^4$ does not depend on normalization

Table 4: String tension parameter $\sqrt{\sigma}$ (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2.

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.2098	0.3398	0.3297	0.3426
0.30	0.2274	0.3615	0.3522	0.3654
0.31	0.2451	0.3827	0.3745	0.3878
0.32	0.2630	0.4035	0.3964	0.4099
0.33	0.2809	0.4239	0.4180	0.4315
0.34	0.2990	0.4437	0.4392	0.4527
0.35	0.3170	0.4629	0.4599	0.4733
0.36	0.3351	0.4817	0.4802	0.4935

conditions for α_s . The values of $K_{\text{fr}}^{1/4}/\Lambda_{QCD}$ calculated with the aid of expression (21) are presented in Table 5. For normalized solutions values of

Table 5: Dimensionless quantities $K_{\text{fr}}^{1/4}/\Lambda_{QCD}$, y_0 , ξ , $K_{\text{an}}^{1/4}/\Lambda_{QCD}$ on loop numbers, $n_f = 3$.

	1-loop	2-loop	3-loop	4-loop
$K_{\text{fr}}^{1/4}/\Lambda_{QCD}$	0.9799	0.8977	0.9952	1.0709
y_0	1.0	0.8299	2.2691	2.8710
ξ	7.3893	9.3702	4.5225	4.3297
$K_{\text{an}}^{1/4}/\Lambda_{QCD}$	0.9373	0.8494	0.9353	1.0025

the gluon condensate $K_{\text{fr}}^{1/4}$ are presented in Table 6. It is seen that the two — four-loop results are quite stable with respect to loop numbers and for normalization condition $\alpha_s(m_\tau^2) = 0.33$ one has $K_{\text{fr}} = (0.355 - 0.383 \text{ GeV})^4$, which is close to the conventional value [14] of the gluon condensate $(0.33 \text{ GeV})^4$.

3.2 Analytic behavior of perturbative component

Let us consider another variant of definition of the perturbative part of α_s for $q^2 < q_0^2$. Namely instead of freezing (20) we shall assume “forced” analytic

Table 6: Values of the gluon condensate $K_{\text{fr}}^{1/4}$ (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2.

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.1568	0.2844	0.2859	0.3038
0.30	0.1699	0.3025	0.3054	0.3240
0.31	0.1832	0.3203	0.3247	0.3439
0.32	0.1965	0.3377	0.3437	0.3635
0.33	0.2099	0.3547	0.3624	0.3827
0.34	0.2234	0.3713	0.3808	0.4014
0.35	0.2369	0.3874	0.3988	0.4198
0.36	0.2504	0.4031	0.4164	0.4377

behavior of α_{pt} in this region,

$$\alpha_{\text{pt}}(q^2) = \alpha_{\text{an}}(q^2), \quad q^2 < q_0^2. \quad (22)$$

The main ideas of the analytic approach in quantum field theory, which allows one to overcome difficulties, connected with nonphysical singularities in perturbative expressions, were proposed in Refs. [15, 16]. The analytic approach is successfully applied to QCD [17]. The forced analytic running coupling constant is defined by the spectral representation

$$a_{\text{an}}(y) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{y + \sigma} \rho(\sigma), \quad (23)$$

where spectral density $\rho(\sigma) = \Im a_{\text{an}}(-\sigma - i0) = \Im a(-\sigma - i0)$. For the perturbative solutions $a(x)$ of the form (13) the two-loop analytic running coupling constant and its nonperturbative part were studied in Ref. [18], the three-loop case and the four-loop case were studied in Refs. [19] and Refs. [20], respectively. Let us write the spectral density up to the four-loop case.

$$\rho^{(1)}(\sigma) = \frac{\pi}{t^2 + \pi^2}, \quad (24)$$

$$\rho^{(2)}(\sigma) = \rho^{(1)}(\sigma) - \frac{b}{(t^2 + \pi^2)^2} [2\pi t F_1(t) - (t^2 - \pi^2) F_2(t)], \quad (25)$$

$$\begin{aligned}\rho^{(3)}(\sigma) = \rho^{(2)}(\sigma) + \frac{b^2}{(t^2 + \pi^2)^3} & \left[\pi (3t^2 - \pi^2) (F_1^2(t) - F_2^2(t)) \right. \\ & - 2t (t^2 - 3\pi^2) F_1(t) F_2(t) - \pi (3t^2 - \pi^2) F_1(t) + t (t^2 - 3\pi^2) F_2(t) \\ & \left. + \pi \kappa (3t^2 - \pi^2) \right],\end{aligned}\quad (26)$$

$$\begin{aligned}\rho^{(4)}(\sigma) = \rho^{(3)}(\sigma) - \frac{b^3}{(t^2 + \pi^2)^4} & \left[(t^4 - 6\pi^2 t^2 + \pi^4) (F_2^3(t) - 3F_1^2(t) F_2(t)) \right. \\ & + 4\pi t (t^2 - \pi^2) (F_1^3(t) - 3F_1(t) F_2^2(t)) - 10\pi t (t^2 - \pi^2) (F_1^2(t) - F_2^2(t)) \\ & + 5 (t^4 - 6\pi^2 t^2 + \pi^4) F_1(t) F_2(t) + 4\pi (1 + 3\kappa) t (t^2 - \pi^2) F_1(t) \\ & \left. - (1 + 3\kappa) (t^4 - 6\pi^2 t^2 + \pi^4) F_2(t) + 4\pi \bar{\kappa} t (t^2 - \pi^2) \right].\end{aligned}\quad (27)$$

Here $t = \ln(\sigma)$,

$$F_1(t) \equiv \frac{1}{2} \ln(t^2 + \pi^2), \quad F_2(t) \equiv \arccos \frac{t}{\sqrt{t^2 + \pi^2}}. \quad (28)$$

Solving the equation²

$$a_{\text{an}}(y) = b_0 h_0, \quad (29)$$

we find values y_0 for a_{an} , being defined by formulas (23) – (28) with the use of values h_0 obtained above. Further we find dimensionless quantity $\xi = (\Lambda_{\text{an}}/\Lambda_{QCD})^2 = x_0/y_0$. Values of y_0 and ξ in dependence on the number of loops are presented in Table 5. In Table 7 values of $\Lambda_{\text{an}} = q_0/\sqrt{y_0}$ are presented in dependence on the number of loops and on normalization conditions at $q^2 = m_\tau^2$.

Let us turn to the gluon condensate. In the considered method of definition of the perturbative part of α_s in the nonperturbative region we have

$$\begin{aligned}K_{\text{an}} = \frac{3}{\pi^3} \int_0^{q_0^2} dq^2 q^2 (\alpha_s(q^2) - \alpha_{\text{an}}(q^2)) &= \frac{12}{\pi^2} \int_0^{q_0^2} dq^2 q^2 \left(\frac{C}{q^2} + z \right. \\ & \left. - \frac{1}{b_0} a_{\text{an}} \left(\frac{q^2}{\Lambda_{\text{an}}^2} \right) \right).\end{aligned}\quad (30)$$

²While solving this equation it is convenient to use the method of Refs. [19], in which $\alpha_{\text{an}}^{\text{npt}}(y)$ is represented as a series in inverse powers of y .

Taking into account the spectral representation (23) and performing integration in Eq. (30), we obtain

$$K_{\text{an}} = \frac{12}{\pi^2} \left(C q_0^2 + \frac{z}{2} q_0^4 \right) - \frac{12 \Lambda_{\text{an}}^4}{\pi^3 b_0} \int_0^\infty d\sigma \rho(\sigma) \left[y_0 - \sigma \ln \left(1 + \frac{y_0}{\sigma} \right) \right]. \quad (31)$$

Substitution $\sigma = \exp(t)$ leads to the following representation of the gluon condensate

$$K_{\text{an}} = \Lambda_{QCD}^4 \left[\frac{12}{\pi^2} \left(h_0 - \frac{z}{2} \right) x_0^2 - \frac{12 \xi^2}{\pi^3 b_0} \int_{-\infty}^\infty dt \rho(t) \left\{ y_0 e^t - e^{2t} \ln(1 + y_0 e^{-t}) \right\} \right]. \quad (32)$$

The expression in the square brackets of (32) does not depend on values of $\alpha_s(m_\tau^2)$, values of the ratio $K_{\text{an}}^{1/4}/\Lambda_{QCD}$, which are obtained by numerical integration in formula (32), are given in Table 5. In Table 7 and Table 8 values of the parameter Λ_{an} and of the gluon condensate $K_{\text{an}}^{1/4}$ are presented, respectively.

Table 7: Dependence of the parameter Λ_{an} (GeV) on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2.

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.4350	0.9696	0.6109	0.5903
0.30	0.4714	1.0315	0.6526	0.6296
0.31	0.5081	1.0921	0.6939	0.6682
0.32	0.5451	1.1515	0.7345	0.7063
0.33	0.5824	1.2094	0.7745	0.7435
0.34	0.6198	1.2659	0.8138	0.7800
0.35	0.6572	1.3210	0.8522	0.8156
0.36	0.6947	1.3745	0.8899	0.8504

For cases from one loop up to four loops the behavior of the running coupling constant $\alpha_s(q^2)$ for all $q^2 > 0$ is shown in Fig. 1. Here the behavior of the analytic coupling constant $\alpha_{\text{an}}(q^2)$ for $q^2 < q_0^2$, which defines the perturbative part of $\alpha_s(q^2)$ in this region, is also shown. In Fig. 2 in addition

Table 8: Values of the gluon condensate $K_{\text{an}}^{1/4}$ (GeV) in dependence on loop numbers and normalization conditions. Normalization conditions are the same as in Table 2.

$\alpha_s(m_\tau^2)$	1-loop	2-loop	3-loop	4-loop
0.29	0.1500	0.2691	0.2687	0.2844
0.30	0.1625	0.2862	0.2870	0.3033
0.31	0.1752	0.3031	0.3052	0.3219
0.32	0.1880	0.3195	0.3230	0.3403
0.33	0.2008	0.3356	0.3406	0.3582
0.34	0.2137	0.3513	0.3579	0.3758
0.35	0.2266	0.3666	0.3748	0.3929
0.36	0.2395	0.3814	0.3914	0.4097

to $\alpha_s(q^2)$ its nonperturbative part $\alpha_{\text{npt}}(q^2)$ is shown, which turns to be zero for $q^2 > q_0^2$. Normalization condition for all curves in Fig. 1, Fig. 2 is $\alpha_s(m_\tau^2) = 0.32$.

4 Conclusion

Starting from the well-known perturbative expression (1) for the β -function for small values of the coupling constant h and the behavior (4) for large values of the coupling constant, which corresponds to the linear confinement, we have constructed the model β -function for all $h > 0$. To control dependence of the results on the number of loops we simultaneously consider cases corresponding to perturbative β -function for 1 — 4 loops.

While constructing the matched β -function we assume it to be smooth at the matching point, that leads to the invariant charge being smooth together with its derivative for all $q^2 > 0$. The normalization of the invariant charge, e.g. at m_τ fix it thoroughly. Then value $\alpha_s(m_\tau^2) \simeq 0.33$ corresponds to value of the parameter $\sigma^{1/2} \simeq 0.42$ GeV of our model which fits the string tension parameter of the string model [21].

The obtained invariant charge is applied to study of the important physical quantity, the gluon condensate. In doing this we consider two variants of extracting of the nonperturbative contributions from the overall expression for the invariant charge. The first variant assumes “freezing” of perturba-

tive part of the charge in the nonperturbative region $q^2 < q_0^2$,³ while for the second one we choose the analytic behavior of the perturbative part in this region. As we see from Tables 6, 8 the first variant leads to values of the gluon condensate being somewhat larger, than that for the second variant.

Emphasize, that for $\alpha_s(m_\tau^2) = 0.33$ values of the gluon condensate for both variants are quite satisfactory. Namely the value for the second variant practically coincides the conventional value [14], while for the first variant it is only slightly higher, $K^{1/4} = K_{\text{fr}}^{1/4} \simeq 0.36$ GeV. Note, that other important parameter, the nonperturbative scale q_0 for the same normalization condition also turns to be of a reasonable magnitude, $q_0 \simeq 1.17$ GeV (see Table 3). We may conclude, that the present model consistently describes the most important nonperturbative parameters. From this point of view the results support a possibility of singular infrared behaviour (3) of the invariant charge which contains both perturbative and nonperturbative contributions.

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³It is seen from Table 1 that freezing occurs at the level $\alpha_s = \alpha_0 \simeq 0.4354$ (average for 3-loop and 4-loop cases) which is very close to critical value $\alpha_{\text{crit}} = 0.137 \cdot \pi = 0.4304$ of Ref. [22].

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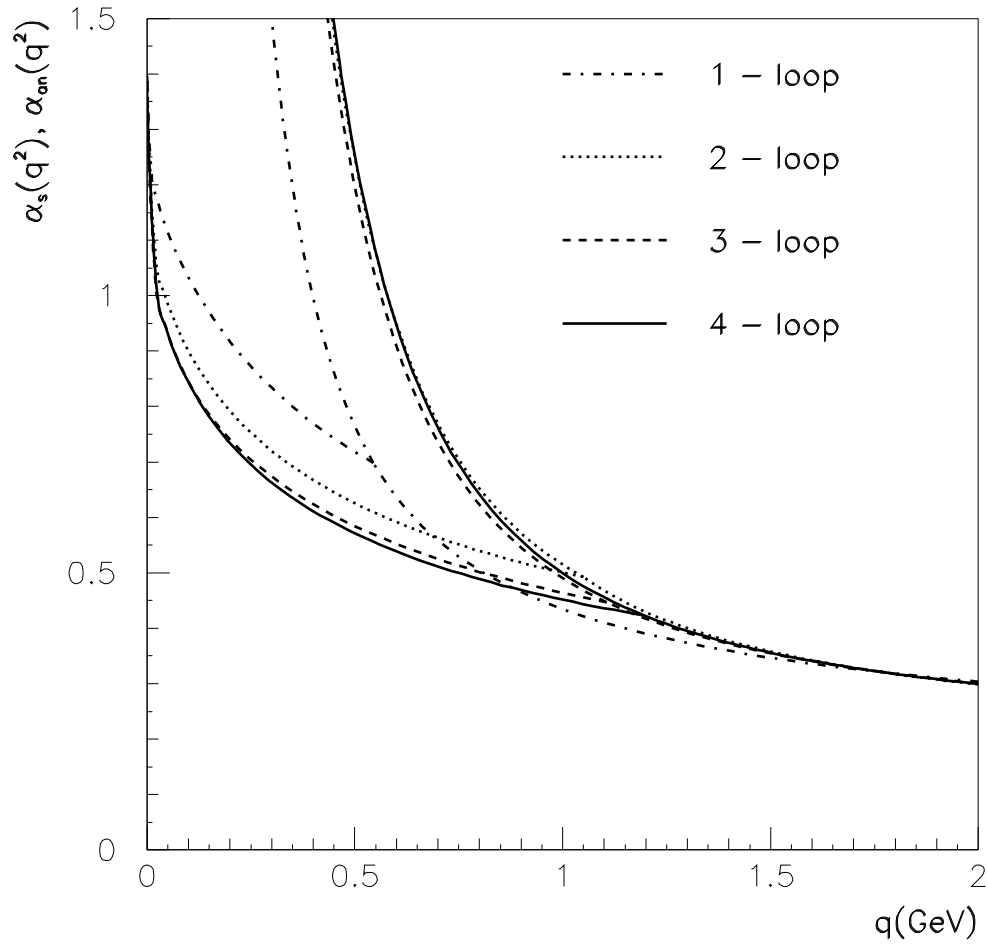


Figure 1: Running coupling constant $\alpha_s(q^2)$ for all $q^2 > 0$ and analytic coupling constant $\alpha_{\text{an}}(q^2)$ for $q^2 < q_0^2$ (the corresponding curves are the lower ones). Normalization conditions: $\alpha_s(m_\tau^2) = 0.32$, $\alpha_{\text{an}}(q_0^2) = \alpha_s(q_0^2)$.

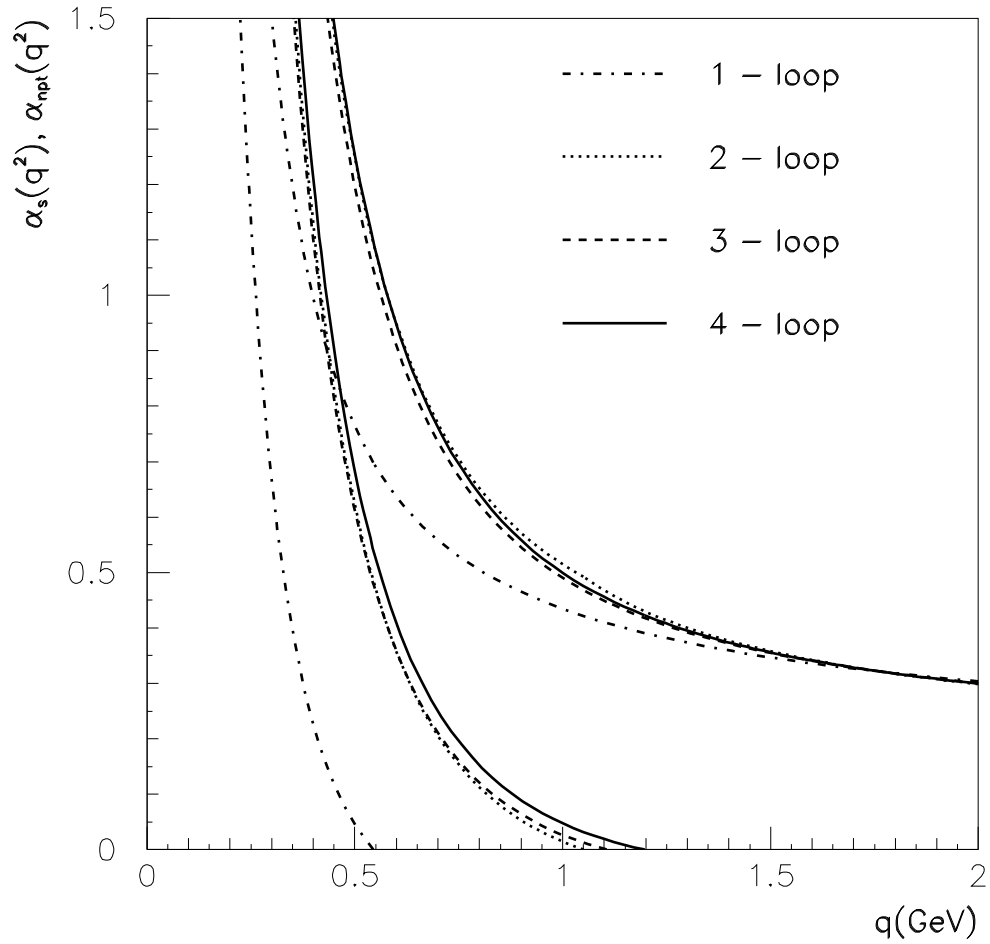


Figure 2: Running coupling constant $\alpha_s(q^2)$ and its nonperturbative part $\alpha_{\text{npt}}(q^2)$ (the corresponding curves are the lower ones) with definition of $\alpha_{\text{pert}}(q^2)$ by (22). Normalization condition: $\alpha_s(m_\tau^2) = 0.32$.